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THE MEMORY GAME

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In contrast to the simplicity of the optimal strategy, the analysis leading to its optimality proof is rather involved. It supplies an interesting example of concrete mathematics of the sort used in the analysis of algorithms. It is doubtful whether this analysis could have been carried out without resort to experimentation and substantial use of automated symbolic computation.

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Abstract

The memory game is a popular card game played by children and adults around the world. Good memory is one of the qualities required in order to succeed in it. This however is not enough. When it is assumed that the players have perfect memory, the memory game can be seen as a game of strategy. The game is analysed under this assumption and the optimal strategy is found. This is simple and perhaps unexpected.

In contrast to the simplicity of the optimal strategy, the analysis leading to its optimality proof is rather involved. It supplies an interesting example of concrete mathematics of the sort used in the analysis of algorithms. It is doubtful whether this analysis could have been carried out without resort to experimentation and substantial use of automated symbolic computation.

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1 The game

A pack containing n pairs of identical cards is shuffled and the cards are spread face down on a table. Each player in turn flips two cards, one after the other. If the two cards flipped are identical (i.e., they form a pair), they are removed from the table into the possession of the player who had flipped them and he/she gets another turn. If the two cards are not identical then they are flipped back and the turn passes to the next player. The game continues until all the cards are removed from the table (or until all the players agree to end the game) and the winner is the player possessing the largest number of pairs. The gain (or loss, if negative) of a player at any stage is defined to be the number of pairs he/she holds minus the average number of pairs held by the opponents.

Any number of players can play the game but the most interesting situation occurs when there are only two of them. We will therefore consider this case here.

The invention of the memory game is sometimes attributed to Christopher Louis Pelman and the game is often called *Pelmanism* (consult this entry in [3]). A description of the game may also be found in [4].

A light-hearted introduction to some of the results in this paper is given in [5].

2 Moves, positions and strategies

Each player tries to remember the position and the identity of all the cards already inspected. To focus our attention on the strategic questions involved we will assume that the players have already reached a high level of proficiency and are able to absorb all this information (in other words, they have perfect memories).

A turn in the game is composed of two plies. In each of them the player has to flip a card. The observation that triggered the present work is that at each ply the player can decide whether to inspect a new card, i.e., a card which was not inspected before (in which case we assume that the outcome is uniformly distributed over all the as-yet-unflipped cards) or to make an idle ply and flip an old card, i.e., a card whose identity is already known to him/her (and his/her opponent). An idle move is not always possible. In the beginning of the game, for example, the first player has to flip two new cards.

There are at most three reasonable moves from each position. The first is to pick no new cards at all. Such a move will be called a *0-move* and it is possible only if there are at least two inspected cards on the table. The two other moves, termed *1-move* and *2-move*, both begin by flipping a new card. If the new card matches a previously inspected card then in both cases the matching card is flipped, a pair is formed and the player gets another turn. If however the first card flipped does not match a previously inspected card then an idle ply is used in a 1-move while a new card is inspected in a 2-move. It can be easily seen that the other possible move, that of making an idle ply first and then flipping a new card is always inferior to the other moves.

While playing the game the players can have two different objectives. They could try to maximise the probability of winning the game or, alternatively, they could try to maximise their expected gain. The two objectives lead to different optimal strategies. We will investigate here the strategy that maximises the expected gain. The optimal strategy for the other case could presumably be obtained using similar methods and more involved analysis.

If a 0-move maximizes the expected gain for the next player then, after this 0-move is played, the situation remains exactly the same and the second player would also like to play a 0-move. Since this can go on for ever we stop the game in such a case.

A position in the game is characterised by the number n of pairs still on the table and the number k of cards which have already been inspected. We can assume that all the inspected cards are different. In the case where the last player played a 2-move and the second card flipped matches not the first card flipped but one of the previously inspected cards, the resulting pair would be immediately removed by the other player, and we may account for this as part of the present turn.

A *strategy* is a rule which determines which one of the three plausible moves should be used in each position (n,k) where $0 \leq k \leq n$ are integers. An *optimal strategy* is a strategy which maximises the expected gain assuming that both players play optimally. The *value* of a position is the expected future gain of the player who is first to play from that position assuming that both players use an optimal strategy. We shall see in the next section that the position values and an optimal strategy can be defined mutually recursively. It is easy to see that if a player is playing according to an optimal strategy then the expected gain from some position is at least the value of that position no matter what strategy the opponent may choose.

3 The optimal strategy

We recursively define the values $e_{n,k}$ of the different positions. The only initial condition that we need is that $e_{0,0} = 0$, that is, that no one gains from a null game. Assume that we have already defined $e_{n',k'}$ for $n' < n$ and $e_{n,k'}$ for $k' > k$. We will first define $e^1_{n,k}$ and $e^2_{n,k}$ which will be the expected gain from position (n,k) when beginning with a 1- or a 2-move respectively, and subsequently playing using an optimal strategy. Consulting Fig. 1 it is relatively easy to verify that

$$\begin{aligned} e^1_{n,k} &= \tfrac{k}{2n-k}(1+e_{n-1,k-1}) - \tfrac{2(n-k)}{2n-k} e_{n,k+1} \quad , \\ e^2_{n,k} &= \tfrac{k}{2n-k}(1+e_{n-1,k-1}) - \tfrac{2(n-k)}{2n-k} \left(\tfrac{k-1}{2n-k-1}(1+e_{n-1,k}) + \tfrac{2(n-k-1)}{2n-k-1} e_{n,k+2} \right) \quad . \end{aligned}$$

We will explain the first relation as an example. When flipping the first card, there are k inspected cards on the table, all of them different, and $2n-k$ uninspected cards. In a 1-move an uninspected card is flipped. With probability $\frac{k}{2n-k}$ it will be a card which

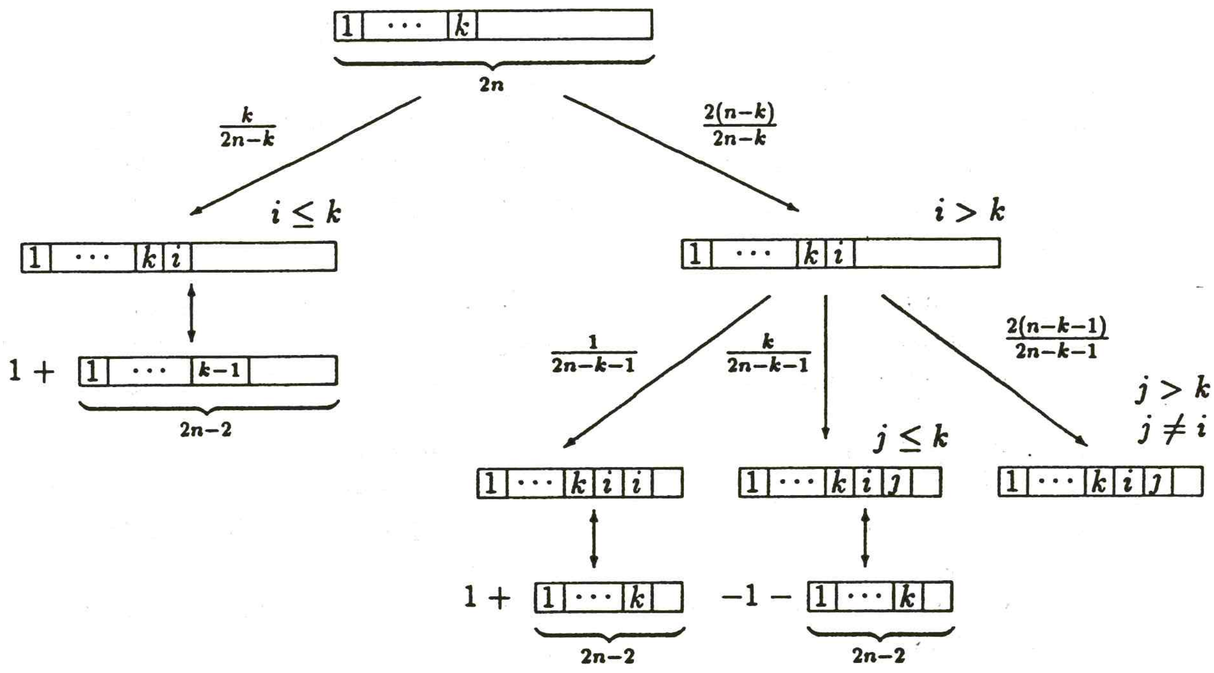

Figure 1: Possible outcomes of moves from position (n, k).

matches one of the previously inspected cards, in which case the player will gain a pair and will be entitled to play again from position $(n-1, k-1)$. With the complementary probability $\frac{2(n-k)}{2n-k}$ the first card flipped will not match any previously inspected card, an idle ply will follow and the opponent will play from position $(n, k+1)$. Since the gain of one player is the other's loss, the expected gain of a player from a position $(n, k+1)$ when the opponent is about to play is $-e_{n,k+1}$. This accounts for the two terms appearing in the first relation. The second relation is obtained in a similar way. (Consulting Fig. 1 may again be useful).

The value $e_{n,k}$ of the position (n, k) with $n > 0$ is now defined as

$$\begin{aligned} e_{n,0} &= e^2_{n,0} \\ e_{n,1} &= \max\{e^1_{n,1}, e^2_{n,1}\} \\ e_{n,k} &= \max\{0, e^1_{n,k}, e^2_{n,k}\} \quad \text{for } 2 \leq k \leq n \quad . \end{aligned}$$

These definitions are explained by the following observations. A 2-move is the only legal move from position $(n, 0)$. A 1-move and a 2-move are the only two moves allowed from position $(n, 1)$. In positions of the form (n, k) where $k \geq 2$ a 0-move could be used. If $e^1_{n,k}, e^2_{n,k} < 0$ then it is advantageous to use a 0-move and the game will stop with value 0.

We say that an i-move is *optimal* from position (n, k) if $e_{n,k} = e^i_{n,k}$. It is possible that more than one move will be optimal from a certain position.

Using these recursive definitions we can compute the values and find the optimal moves. Table 1 gives the values of positions with $n \leq 7$ while Table 2 gives the optimal moves for

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$n=0$	0							
$n=1$	1	1						
$n=2$	$-\frac{2}{3}$	$\frac{2}{3}$	2					
$n=3$	$-\frac{1}{5}$	$-\frac{1}{5}$	$\frac{1}{3}$	3				
$n=4$	$-\frac{4}{35}$	$\frac{4}{35}$	$\frac{4}{15}$	0	4			
$n=5$	$-\frac{1}{35}$	$-\frac{1}{35}$	$\frac{1}{7}$	$\frac{19}{35}$	0	5		
$n=6$	$\frac{2}{1155}$	$\frac{2}{1155}$	$\frac{2}{21}$	$\frac{13}{105}$	$\frac{27}{35}$	0	6	
$n=7$	$\frac{61}{1155}$	$\frac{61}{1155}$	$\frac{13}{495}$	$\frac{53}{231}$	$\frac{2}{21}$	$\frac{62}{63}$	0	7

Table 1: The expected values of the simplest positions

$n \leq 15$. For $(n,k) = (4,3)$ it turns out that both the 0-move and the 2-move are optimal but only the 2-move is listed in the table. Similarly, for any n, $e^1_{n,n} = e^2_{n,n} = n$, so both the 1-move and the 2-move are optimal in this case. In fact, they are identical in this case since the first card flipped will always match a previously inspected card.

The pattern emerging from Table 2 is clear. A 2-move should be used when $k = 0$, since this is the only allowed move. A 1-move should be used whenever $k > 0$ and $n + k$ is even. Either a 2-move or a 0-move should be used when $n+k$ is odd ($(n,k) = (6,1)$ being the only exception). Inspecting a few more rows in the table immediately suggests that a 0-move should be used when in addition to the requirement that $n + k$ is odd we also have $k \geq 2(n+1)/3$.

We thus claim :

Theorem 3.1

$$e_{n,k} = \begin{cases} 0 & \text{if } [k \geq \frac{2(n+1)}{3} \text{ and } n+k \text{ odd}] \\ e^1_{n,k} & \text{if } [k \geq 1 \text{ and } n+k \text{ even}] \text{ or } [(n,k) = (6,1)] \\ e^2_{n,k} & \text{otherwise.} \end{cases}$$

Another interesting issue is the behaviour of the values $e_{n,k}$ themselves. The following approximation gives their asymptotical behaviour.

Theorem 3.2

$$e_{n,k} = \begin{cases} \frac{k}{2(n-k)+1} + O(n^{-1}) & \text{if } n+k \text{ even} \\ \frac{(2n-3k)(2n-k)}{16(n-k)^3} + O(n^{-2}) & \text{if } n+k \text{ odd and } k \leq \frac{2n+1}{3} \\ 0 & \text{if } n+k \text{ odd and } k \geq \frac{2(n+1)}{3} \end{cases} .$$

$n=1$	2 1
$n=2$	2 2 1
$n=3$	2 1 2 1
$n=4$	2 2 1 2 1
$n=5$	2 1 2 1 0 1
$n=6$	2 1 1 2 1 0 1
$n=7$	2 1 2 1 2 1 0 1
$n=8$	2 2 1 2 1 2 1 0 1
$n=9$	2 1 2 1 2 1 2 1 0 1
$n=10$	2 2 1 2 1 2 1 2 1 0 1
$n=11$	2 1 2 1 2 1 2 1 0 1 0 1
$n=12$	2 2 1 2 1 2 1 2 1 0 1 0 1
$n=13$	2 1 2 1 2 1 2 1 2 1 0 1 0 1
$n=14$	2 2 1 2 1 2 1 2 1 2 1 0 1 0 1
$n=15$	2 1 2 1 2 1 2 1 2 1 2 1 0 1 0 1

Table 2: The optimal moves for $n \leq 15$.

If we let $\lambda = k/n$ then we see that for $\lambda < 1$, $e_{n,k} = e^1_{n,k} \sim \frac{\lambda}{2(1-\lambda)}$ if $n+k$ is even, and $e_{n,k} = e^2_{n,k} \sim \frac{(2-3\lambda)(2-\lambda)}{16(1-\lambda)^3} \cdot \frac{1}{n}$ if $n+k$ is odd and $k \leq \frac{2n+1}{3}$. Similarly, we can get that $e^2_{n,k} \sim -\frac{\lambda}{2(1-\lambda)} \cdot \frac{4-12\lambda+7\lambda^2}{(2-\lambda)^2}$ if $n+k$ is even and that $e^1_{n,k} = -\frac{4-8\lambda+5\lambda^2}{16(1-\lambda)^3} \cdot \frac{1}{n}$ if $n+k$ is odd and $k \leq \frac{2n+1}{3}$.

A graph of $e_{n,k} = e^1_{n,k}$ and $e^2_{n,k}$ for even values of $n+k$ is given in Fig. 2. It can be seen that, unless λ is very small, $e^1_{n,k}$ is both positive and markedly superior to $e^2_{n,k}$. Similarly, a graph contrasting $e_{n,k} = e^2_{n,k}$ with $e^1_{n,k}$ for odd values of $n+k$ is presented in Fig. 3. Again there is a sharp difference between these two options.

However we need better approximations to show that $e_{n,k} = e^1_{n,k}$ when $n+k$ is even and $k = o(n)$, and that $e_{n,k} = e^2_{n,k}$ when $n+k$ is odd and $k = \frac{2-o(n)}{3}$. These are obtained in the next section.

Another interesting question is the sign of the values of the different positions. By definition $e_{n,k} \geq 0$ whenever $k \geq 2$, but what happens when $k=1$ or $k=0$? In particular, when is it advantageous to take the first turn? It turns out that $e_{n,1} > 0$ for $n \geq 6$, that $e_{n,0} > 0$ when $n \geq 7$ and n is odd, and that $e_{n,0} < 0$ when $n \geq 8$ and n is even. Thus it is advantageous to take the first move in the game if and only if n is either $1, 6$ or an odd number greater or equal to 7.

Finally, what is the expected gain or loss from a game played with n pairs of cards? It turns out that, for large n, the gain or loss is roughly $1/4n$. More precisely

Theorem 3.3

$$e_{n,0} = \begin{cases} \frac{1}{4n+2} + O(\frac{1}{n^3}) & \textit{if } n \textit{ odd} \\ -\frac{1}{4n-2} + O(\frac{1}{n^3}) & \textit{if } n \textit{ even.} \end{cases}$$

The proofs of the theorems stated in this section are given in the next section.

4 Analysis

Our strategy for proving the results claimed in the previous section is the following. We first investigate the expected gains from each position when the two players play according to the alleged optimal strategy. Once we have tight estimations of these values it will be easy to prove by induction that these values do in fact correspond to the optimal strategy.

4.1 Preliminary manipulations

Let $e_{n,k}$ be the expected values of the different positions when both players play according to the conjectured optimal strategy. As a 'warm-up' we prove the following lemma

Lemma 4.1 *(i)* $e_{n,0} = e_{n,1}$ *for odd* $n \geq 1$,
(ii) $e_{n,0} = -e_{n,1}$ *for even* $n \neq 6$.

Proof : For odd n, we have $e_{n,0} = e^2_{n,0}$ and $e_{n,1} = e^1_{n,1}$. Consulting the definitions of $e^2_{n,k}$ and $e^1_{n,k}$ from Section 3, we see that both $e^2_{n,0}$ and $e^1_{n,1}$ expand to the same expression

$$e^2_{n,0} = e^1_{n,1} = \tfrac{1}{2n-1}(1 + e_{n-1,0}) - \tfrac{2(n-1)}{2n-1}e_{n,2} \quad .$$

For even $n \neq 6$, we have $e_{n,0} = e^2_{n,0}$ and $e_{n,1} = e^2_{n,1}$ and therefore

$$\begin{aligned} e_{n,0} + e_{n,1} &= \left[\tfrac{1}{2n-1}(1 + e_{n-1,0}) - \tfrac{2(n-1)}{2n-1}e_{n,2}\right] + \left[\tfrac{1}{2n-1}(1 + e_{n-1,0}) - \tfrac{2(n-1)}{2n-1} \cdot \tfrac{2(n-2)}{2n-2}e_{n,2}\right] \\ &= \tfrac{2}{2n-1}(1 + e_{n-1,0}) - \tfrac{2(n-1)}{2n-1}e_{n,2} - \tfrac{2(n-2)}{2n-1}e_{n,3} \quad . \end{aligned}$$

For even $n \geq 2$, we have $e_{n,2} = e^1_{n,2}$ and thus

$$\begin{aligned} e_{n,0} + e_{n,1} &= \tfrac{2}{2n-1}(1 + e_{n-1,0}) - \tfrac{2(n-1)}{2n-1}\left[\tfrac{2}{2n-2}(1 + e_{n-1,1}) - \tfrac{2(n-2)}{2n-2}e_{n,3}\right] - \tfrac{2(n-2)}{2n-1}e_{n,3} \\ &= \tfrac{2}{2n-1}[e_{n-1,0} - e_{n-1,1}] = 0 \end{aligned}$$

where the last equality follows from the first part of the Lemma. □

As an easy corollary we get

Lemma 4.2 $e^1_{n,0} = e^2_{n,0}$ *for even* $n \neq 6$.

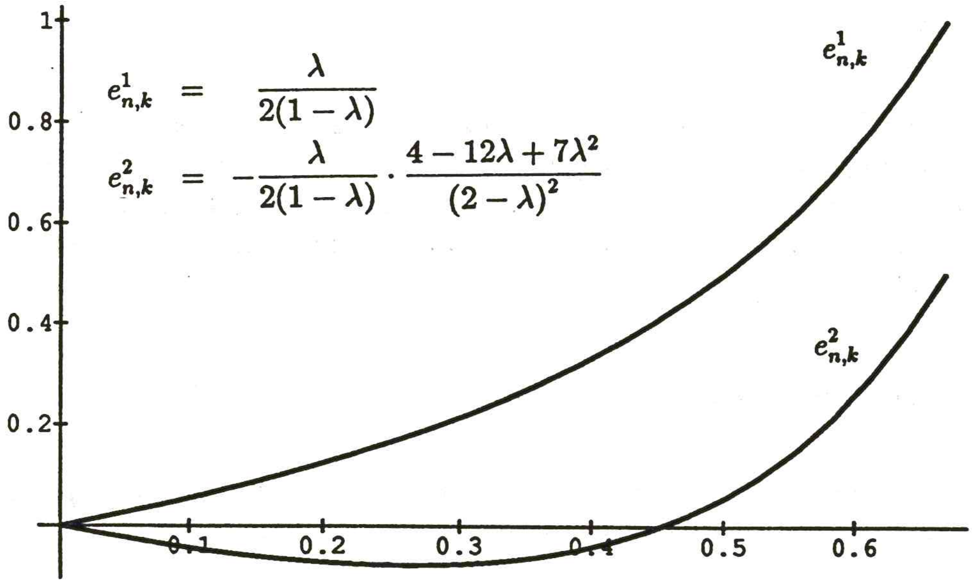

Figure 2: The behaviour of $e_{n,k} = e^1_{n,k}$ and $e^2_{n,k}$ for $n+k$ even.

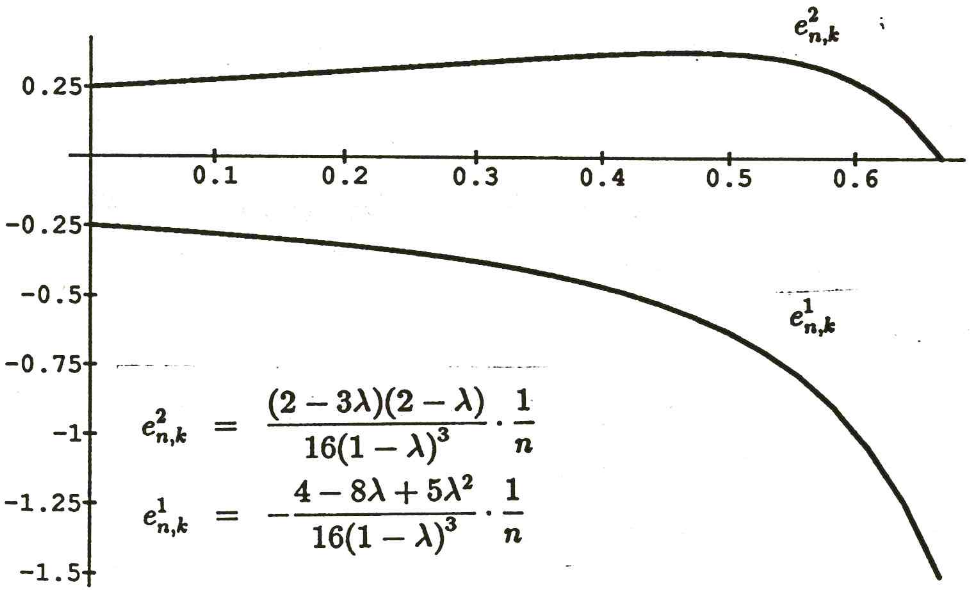

Figure 3: The behaviour of $e_{n,k} = e^2_{n,k}$ and $e^1_{n,k}$ for $n+k$ odd.

Proof : By the definition we have $e^2_{n,0} = e_{n,0}$ and $e^1_{n,0} = -e_{n,1}$ and thus the result follows from the second part of the previous Lemma. $\square$

Note that 1-moves are currently not allowed from positions of the form $(n, 0)$. The previous Lemma says however that it would not matter if we were to allow them from these positions with even $n \neq 6$. Furthermore, the 1-moves would be co-optimal in these positions and we could therefore use the relation $e_{n,0} = e^1_{n,0}$ as the defining relation for even $n \neq 6$. This removes the anomaly of the column $k = 0$ seen in Table 1. The two remaining exceptions are $e_{6,0} = e^2_{6,0}$ and $e_{6,1} = e^1_{6,1}$.

Since the parity of $n+k$ plays a major role in the following analysis, it will be convenient to denote $e_{n,k}$ by $a_{n,k}$ when $n + k$ is even, and by $b_{n,k}$ when $n + k$ is odd. It is also convenient to write the recurrence relations defining $a_{n,k}$ and $b_{n,k}$ with the help of an auxiliary sequence $c_{n,k}$ as follows

$$\begin{aligned}
a_{n,k} &= p_{n,k}(1 + a_{n-1,k-1}) - q_{n,k}b_{n,k+1} \\
b_{n,k} &= [p_{n,k}(1 + b_{n-1,k-1}) - q_{n,k}c_{n,k+1}] \times I_{n,k} \\
c_{n,k} &= p'_{n,k}(1 + a_{n-1,k-1}) + q_{n,k}b_{n,k+1}
\end{aligned}$$

where

$$p_{n,k} = \tfrac{k}{2n-k} \quad , \quad p'_{n,k} = \tfrac{k-2}{2n-k} \quad , \quad q_{n,k} = \tfrac{2(n-k)}{2n-k}$$

and

$$I_{n,k} = \begin{cases} 1 & \text{if } k < \frac{2n+1}{3} \\ 0 & \text{otherwise} \end{cases} \quad .$$

These relations hold for any (n, k) with the exception of $(6, 0)$ and $(6, 1)$. The only initial condition required is that $a_{0,0} = 0$.

Note that $c_{n,k+1}$ corresponds to the expected loss from position (n, k) if one new card had already been flipped and did not match any of the previously inspected cards.

4.2 Operator notation

The following analysis is facilitated by introducing operator notation. Define an operator Φ by

$$\Phi\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} \quad \text{where} \quad \begin{aligned} a'_{n,k} &= p_{n,k}a_{n-1,k-1} - q_{n,k}b_{n,k+1} \\ b'_{n,k} &= p_{n,k}b_{n-1,k-1} - q_{n,k}c_{n,k+1} \\ c'_{n,k} &= p'_{n,k}a_{n-1,k-1} + q_{n,k}b_{n,k+1} \end{aligned}$$

and an operator Z by

$$Z\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b' \\ c \end{pmatrix} \quad \text{where} \quad b'_{n,k} = \begin{cases} b_{n,k} & \text{if } k \leq \frac{2n+1}{3} \\ 0 & \text{otherwise} \end{cases} \quad .$$

We again assume that Φ has the anomalous behaviour

$$a'_{6,0} = b'_{6,1} = \tfrac{1}{11}(1 + b_{5,0}) - \tfrac{10}{11}a_{6,2} \quad .$$

If we let $e = (a,b,c)^T$ and $h = (p,p,p')^T$ then it is easy to see that e satisfies the following equation

$$\boxed{e = Z(\Phi e + h)} \quad . \tag{1}$$

Our task is to solve this operator equation.

4.3 Bootstrapping

We start by trying to solve the equation obtained by ignoring the presence of the operator Z in equation (1)

$$\boxed{e = \Phi e + h} \quad . \tag{2}$$

The solution of this equation will not only give us some useful information about the solution of equation (1), it also has some interest in its own right. It corresponds to the analysis of the variant of the game in which 0-moves are not allowed.

Solving equation (2) amounts to inverting the operator $(I - \Phi)$, which does not seem an easy task. We approach this by approximating Φ by an operator $\widehat{\Phi}$ for which inverting $(I - \widehat{\Phi})$ is much easier. Using a method that bears some resemblance to the 'bootstrapping' method described in [1], we define a sequence of refining terms $E^0, E^1, \ldots$ whose sum $E^0 + E^1 + \ldots$ converges, we hope, to the required solution. This sequence is obtained in the following way

$$\boxed{\begin{array}{ll} E^i = (I - \widehat{\Phi})^{-1} h^i & , \ i \geq 0 \\ h^{i+1} = (\Phi - \widehat{\Phi}) E^i & , \ i \geq 0 \end{array}}$$

where $h^0 = h$. Let $e^0 = e$, and $e^i = e^{i-1} - E^{i-1}$ for $i \geq 1$ be the error of the i'th approximation. It is easy to verify that

$$\boxed{e^i = \Phi e^i + h^i \quad , \ i \geq 0} \quad .$$

We define $\widehat{\Phi}$ as follows

$$\widehat{\Phi} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} \quad \text{where} \quad \begin{array}{l} a'_{n,k} = p_{n,k} a_{n,k} - q_{n,k} b_{n,k} \\ b'_{n,k} = p_{n,k} b_{n,k} - q_{n,k} c_{n,k} \\ c'_{n,k} = p_{n,k} a_{n,k} + q_{n,k} b_{n,k} \end{array}$$

or equivalently

$$\widehat{\Phi} = \begin{pmatrix} p & -q & 0 \\ 0 & p & -q \\ p & q & 0 \end{pmatrix} \quad .$$

Thus

$$(I-\widehat{\Phi}) = \begin{pmatrix} 1-p & q & 0 \\ 0 & 1-p & q \\ -p & -q & 1 \end{pmatrix} \quad , \quad (I-\widehat{\Phi})^{-1} = \frac{1}{2q^2}\begin{pmatrix} 1+q & -1 & q \\ -p & 1 & -q \\ p & q-p & q \end{pmatrix}$$

and

$$(\Phi-\widehat{\Phi})\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} \quad \text{where} \quad \begin{array}{l} a'_{n,k} = p_{n,k}(a_{n-1,k-1} - a_{n,k}) - q_{n,k}(b_{n,k+1} - b_{n,k}) \\ b'_{n,k} = p_{n,k}(b_{n-1,k-1} - b_{n,k}) - q_{n,k}(c_{n,k+1} - c_{n,k}) \\ c'_{n,k} = (p'_{n,k}a_{n-1,k-1} - p_{n,k}a_{n,k}) + q_{n,k}(b_{n,k+1} - b_{n,k}) \end{array} \quad .$$

The terms E^i obtained in this way become horrendously complicated even for very small values of i and it seems almost impossible to handle them manually. We used *Mathematica* to do these computations.

We now note that for $k < \lambda n$, for some $\lambda < 1$ we have $h^0 = O(1)$ and thus it can easily be seen that $E^0 = O(1)$. The operator $\Phi - \widehat{\Phi}$ has the characteristics of a discrete difference operator. Since each component of $E^0 = (A^0, B^0, C^0)^T$ is a rational function in n and k and thus continuous, in the sense that $A^0_{n-1,k-1} - A^0_{n,k} = O(n^{-1})$ and so on, it is easy to see that $h^1 = O(n^{-1})$. By induction, we can prove in this way that as long as $\lambda = k/n$ is bounded away from 1, we have $E^i = O(n^{-i})$. Therefore each additional term E^i that we compute allows us to obtain an additional term in the asymptotic expansions of a,b and c. These computations can again be done using *Mathematica* and the expansions obtained are

$$a_{n,k} = \frac{\lambda}{2(1-\lambda)} + \frac{\lambda^2+4\lambda-4}{16(1-\lambda)^3}\cdot\frac{1}{n} + \frac{2\lambda^3-4\lambda^2+5\lambda-2}{16(1-\lambda)^5}\cdot\frac{1}{n^2} + \frac{13\lambda^4-62\lambda^3+112\lambda^2-64\lambda+8}{64(1-\lambda)^7}\cdot\frac{1}{n^3} + \dots$$

$$b_{n,k} = \frac{(2-3\lambda)(2-\lambda)}{16(1-\lambda)^3}\cdot\frac{1}{n} + \frac{4\lambda^3-14\lambda^2+11\lambda-2}{16(1-\lambda)^5}\cdot\frac{1}{n^2} + \frac{9\lambda^4-12\lambda^3-60\lambda^2+80\lambda-24}{64(1-\lambda)^7}\cdot\frac{1}{n^3} + \dots$$

The expansion of $c_{n,k}$ is easily obtained from these two. Since it does not have any interest of its own, we do not show it here.

We claim that by truncating these expansions after the $O(n^{-i})$ terms we get an approximation to the solution $e = (a,b,c)^T$ of (2) with errors of $O(n^{-(i+1)})$. In particular, if we let

$$\begin{aligned} A_{n,k} &= \frac{\lambda}{2(1-\lambda)} + \frac{\lambda^2+4\lambda-4}{16(1-\lambda)^3}\cdot\frac{1}{n} + \frac{2\lambda^3-4\lambda^2+5\lambda-2}{16(1-\lambda)^5}\cdot\frac{1}{n^2} \\ B_{n,k} &= \frac{(2-3\lambda)(2-\lambda)}{16(1-\lambda)^3}\cdot\frac{1}{n} + \frac{4\lambda^3-14\lambda^2+11\lambda-2}{16(1-\lambda)^5}\cdot\frac{1}{n^2} \\ C_{n,k} &= \frac{\lambda}{2(1-\lambda)} - \frac{3\lambda^2-16\lambda+12}{16(1-\lambda)^3}\cdot\frac{1}{n} + \frac{14-41\lambda+36\lambda^2-8\lambda^3}{16(1-\lambda)^5}\cdot\frac{1}{n^2} \end{aligned} \qquad (3)$$

where as usual $\lambda = k/n$, we claim that for any $\lambda < c < 1$, where c is constant, we have $a_{n,k} = A_{n,k} + O(n^{-3})$, $b_{n,k} = B_{n,k} + O(n^{-3})$ and $c_{n,k} = C_{n,k} + O(n^{-3})$.

Furthermore, we prove in this section that these expansions are also valid for the solution $e = (a,b,c)^T$ of (1), corresponding to the full version of the game, provided that λ is less than and bounded away from $2/3$. We thus see that in this region there is hardly any difference between the two variants of the game.

4.4 Boundary layer influence

In this sub-section we return to the study of equation (1) that corresponds to the full version of the game.

Let Φ^* be the operator defined as follows

$$\Phi^* \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} \quad \text{where} \quad \begin{aligned} a'_{n,k} &= \begin{cases} p_{n,k}a_{n-1,k-1} - q_{n,k}b_{n,k+1} & \text{if} \quad k \le \frac{2n-2}{3} \\ p_{n,k}a_{n-1,k-1} & \text{if} \quad k \ge \frac{2n-1}{3} \end{cases} \\ b'_{n,k} &= \begin{cases} p_{n,k}b_{n-1,k-1} - q_{n,k}c_{n,k+1} & \text{if} \quad k \le \frac{2n+1}{3} \\ 0 & \text{if} \quad k \ge \frac{2n+2}{3} \end{cases} \\ c'_{n,k} &= \begin{cases} p'_{n,k}a_{n-1,k-1} + q_{n,k}b_{n,k+1} & \text{if} \quad k \le \frac{2n-2}{3} \\ p'_{n,k}a_{n-1,k-1} & \text{if} \quad k \ge \frac{2n-1}{3} \end{cases} \end{aligned} .$$

It is easy to verify that $Z\Phi^* = \Phi^*$ and that if $e = Ze$ then $\Phi e = \Phi^* e$. If we let $h' = Zh$ we therefore get that equation (1) is equivalent to

$$\boxed{e = \Phi^* e + h'} \quad . \tag{4}$$

Examining this equation, we see that the values of $a_{n,k}$ for $k \le \frac{2n+3}{3}$, the values of $b_{n,k}$ for $k \le \frac{2n+1}{3}$, and the values of $c_{n,k}$ for $k \le \frac{2n+4}{3}$ do not depend on values outside these regions. We denote this 'closed' region by Ω and consider the behaviour of e on it first.

The values of $a_{n,k}$ for $\frac{2n-1}{3} \le k \le \frac{2n+3}{3}$, of $b_{n,k}$ for $\frac{2n-4}{3} \le k \le \frac{2n+1}{3}$ and of $c_{n,k}$ for $\frac{2n-1}{3} \le k \le \frac{2n+4}{3}$ are the values in Ω affected most directly by the vanishing of the $b_{n,k}$ terms for $k \ge \frac{2n+2}{3}$. We call the narrow region of Ω containing these values the *boundary layer* of Ω and denote it by $\partial\Omega$. It is convenient to refer to the differences between the actual values $a_{n,k}$,$b_{n,k}$ and $c_{n,k}$ in Ω and those predicted by the asymptotic expansion of the previous sub-section as caused by this boundary layer. The shapes of the region Ω and the boundary layer $\partial\Omega$ are depicted in Fig. 4.

Note that on $\Omega - \partial\Omega$ the operators Φ and Φ^* agree while on $\partial\Omega$ the operator Φ^* has missing $\pm q_{n,k}b_{n,k+1}$ terms. Since in the boundary layer $b_{n,k+1} = O(n^{-2})$ (or more precisely $B_{n,k+1} = O(n^{-2})$), we expect the boundary layer to have only an $O(n^{-2})$ influence on values close to the boundary layer. We shall further see that this influence fades very quickly as we move away from the boundary area.

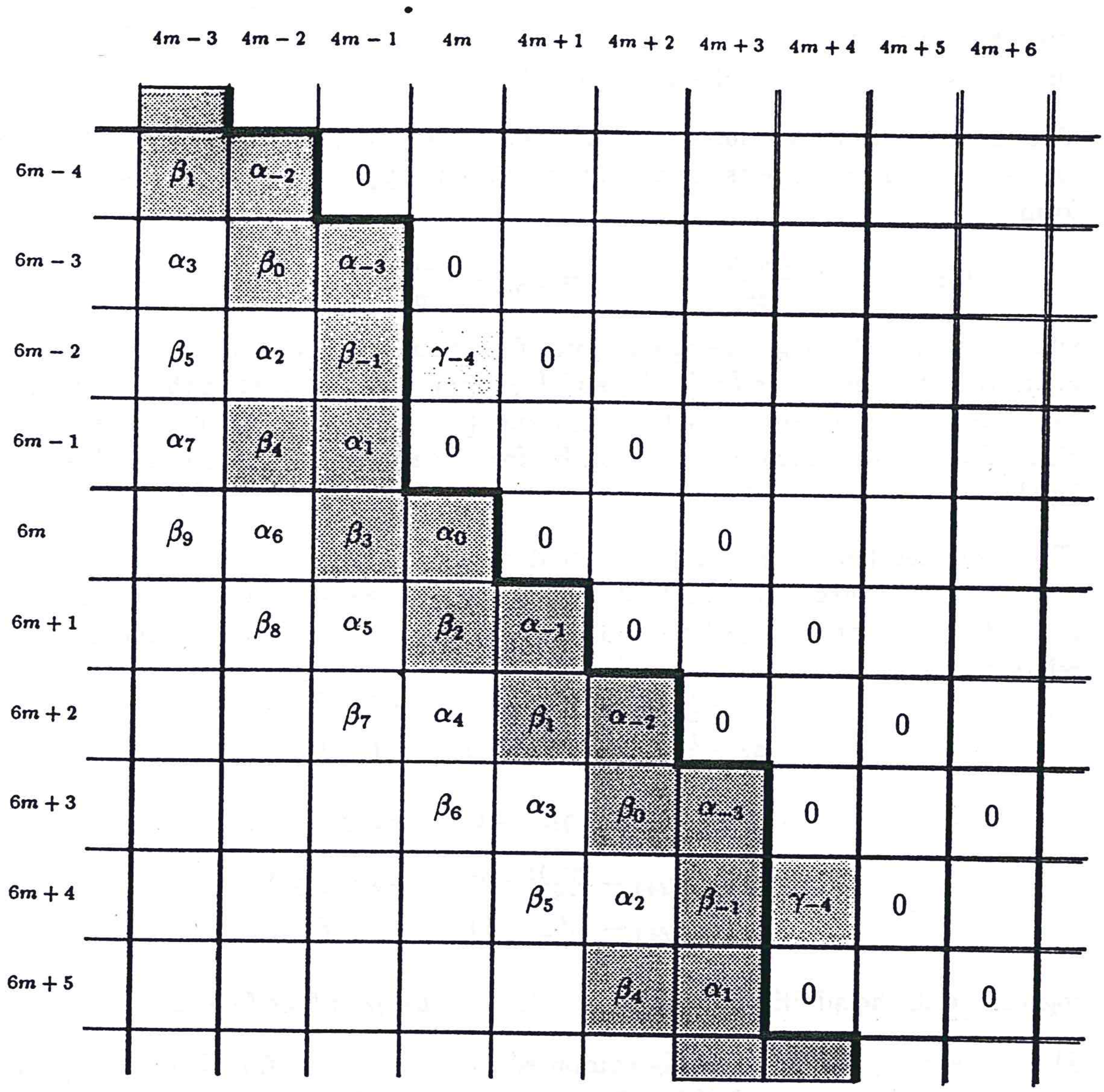

Figure 4: The region Ω and the boundary layer $\partial\Omega$.

We will now try to find an approximation $\mathcal{E}$ with $O(n^{-3})$ error for the solution of (4) valid for the whole of Ω. This approximation will enable us to establish in sub-section 4.7 the validity of the alleged optimal strategy. As implied by the previous paragraph, this approximation must include not only the first terms obtained by the bootstrapping process but also terms corresponding to the boundary layer influence.

If $e = \Phi^* e + h'$ and $\varepsilon = e - \mathcal{E} = O(n^{-3})$ in Ω, then we must also have $\mathcal{H} = (I - \Phi^*)(e - \mathcal{E}) = h' - (I - \Phi^*)\mathcal{E} = O(n^{-3})$ in Ω. Note that ε satisfies the equation $\varepsilon = \Phi^*\varepsilon + \mathcal{H}$. In the next sub-section we will see that under certain conditions, the last implication can be reversed. More precisely, if $\mathcal{H} = O(n^{-3})$ and if it satisfies a certain additional condition then $\varepsilon = e - \mathcal{E} = O(n^{-3})$.

Let us first look at $H = (R, S, T)^T = h' - (I - \Phi^*)E$ where $E = (A, B, C)^T$ with the A, B, C defined in (3). Easy manipulations show that $R_{n,k}, S_{n,k}, T_{n,k} = O(n^{-3})$ in $\Omega - \partial\Omega$

(this is ensured by the bootstrapping process) but that $R_{n,k} = \frac{9(2n-3k-1)}{8} \cdot \frac{1}{n^2} + O(n^{-3})$ and $T_{n,k} = -\frac{9(2n-3k-1)}{8} \cdot \frac{1}{n^2} + O(n^{-3})$ in $\partial\Omega$.

The quantity $2n-3k$ measures the horizontal distance of position (n,k) from the boundary layer $\partial\Omega$. This suggests trying to work with an approximation $\mathcal{E} = (\mathcal{A},\mathcal{B},\mathcal{C})^T$ of the form

$$\mathcal{A}_{n,k} = A_{n,k} + \frac{\alpha_{2n-3k}}{n^2} \quad , \quad \mathcal{B}_{n,k} = B_{n,k} + \frac{\beta_{2n-3k}}{n^2} \quad , \quad \mathcal{C}_{n,k} = C_{n,k} + \frac{\gamma_{2n-3k}}{n^2} \tag{5}$$

where the $A_{n,k}, B_{n,k}, C_{n,k}$ are again those of (3) and thus represent the global behaviour in Ω, while the sequences $\{\alpha_\ell\}$,$\{\beta_\ell\}$ and $\{\gamma_\ell\}$ represent the effect of the boundary layer $\partial\Omega$. We expect the sequences $\{\alpha_\ell\}$,$\{\beta_\ell\}$ and $\{\gamma_\ell\}$ to be quickly, in fact exponentially, diminishing so that their contribution far from the boundary layer will indeed be negligible.

The sequences $\{\alpha_\ell\}$,$\{\beta_\ell\}$ and $\{\gamma_\ell\}$ should be chosen in a way that ensures that $\mathcal{H} = O(n^{-3})$ in the whole of Ω. To that end, as we shall see shortly in the proof of Theorem 4.3, the sequences $\{\alpha_\ell\}$,$\{\beta_\ell\}$ and $\{\gamma_\ell\}$ should satisfy the following linear recurrence relations

$$\begin{array}{lcl}
\alpha_\ell - \frac{1}{2}\alpha_{\ell+1} - \frac{9(\ell-1)}{8} = 0 & , & -3 \le \ell \le 1 \\
\alpha_\ell - \frac{1}{2}\alpha_{\ell+1} + \frac{1}{2}\beta_{\ell-3} = 0 & , & 1 < \ell \\
 & & \\
\beta_\ell - \frac{1}{2}\beta_{\ell+1} + \frac{1}{2}\gamma_{\ell-3} = 0 & , & -1 \ge \ell \\
 & & \\
\gamma_\ell - \frac{1}{2}\alpha_{\ell+1} - \frac{9(\ell-1)}{8} = 0 & , & -4 \le \ell \le 1 \\
\gamma_\ell - \frac{1}{2}\alpha_{\ell+1} - \frac{1}{2}\beta_{\ell-3} = 0 & , & 1 < \ell
\end{array} \tag{6}$$

together with the additional requirement that $\alpha_\ell, \beta_\ell, \gamma_\ell \to 0$ as $\ell \to \infty$.

The values of α_ℓ and β_ℓ are easily computed using generating function techniques. The first values are $\alpha_{-3} \simeq -6.83199877$, $\alpha_{-2} \simeq -4.66399755$, $\alpha_{-1} \simeq -2.57799510$ and $\beta_{-1} \simeq -2.08745613$, $\beta_0 \simeq -1.96591166$, $\beta_1 = -1.76382209$. In general $\alpha_\ell = \sum_{i=1}^{6} u_i\theta_i^{-\ell}$, $\beta_\ell = \sum_{i=1}^{6} v_i\theta_i^{-\ell}$ and $\gamma_\ell = \sum_{i=1}^{6} w_i\theta_i^{-\ell}$ where u_i, v_i, w_i are some fixed complex numbers and $\theta_1,\ldots,\theta_6$ are the six complex roots of the equation $x^8 - x^7 + 4x^2 - 4x + 1 = 0$ with modulus greater than 1. The values of the roots θ_i and of the coefficients u_i and v_i are given in Table 3.

Assuming that $\mathcal{E}$ does indeed approximate e to within an $O(n^{-3})$ error we get (for fixed values of ℓ) the following behaviour of $a_{n,k}$ and $b_{n,k}$ near the boundary layer

$$\begin{array}{rcl}
a_{n,\frac{2n-\ell}{3}} & = & 1 - \frac{3(\ell+1)}{2} \cdot \frac{1}{n} + \left(\frac{3(2\ell^2+2\ell+3)}{4} + \alpha_\ell\right) \cdot \frac{1}{n^2} + O(\frac{1}{n^3}) \quad , \\
b_{n,\frac{2n-\ell}{3}} & = & \left(\frac{9(\ell+1)}{4} + \beta_\ell\right) \cdot \frac{1}{n^2} + O(\frac{1}{n^3}) \quad .
\end{array}$$

In particular

$$b_{n,\frac{2n+1}{3}} \simeq \frac{0.162544}{n^2} \quad , \quad b_{n,\frac{2n}{3}} \simeq \frac{2.534088}{n^2} \quad , \quad b_{n,\frac{2n-1}{3}} \simeq \frac{4.986178}{n^2} \quad .$$

$\theta_{1,2} \simeq -1.108812 \pm 0.625391i$	$u_{1,2} \simeq 0.021830 \mp 0.048470i$	$v_{1,2} \simeq 0.060084 \pm 0.035674i$
$\theta_{3,4} \simeq 1.121061 \pm 0.562315i$	$u_{3,4} \simeq -1.027472 \mp 0.663188i$	$v_{3,4} \simeq -0.227399 \pm 1.791415i$
$\theta_{5,6} \simeq -0.018515 \pm 1.239618i$	$u_{5,6} \simeq -0.122426 \pm 0.028318i$	$v_{5,6} \simeq -0.015567 \pm 0.142098i$
$\theta_7 \simeq 0.539036$	$u_7 = 0.000000$	$v_7 = 0.000000$
$\theta_8 \simeq 0.473498$	$u_8 = 0.000000$	$v_8 = 0.000000$

Table 3: The values of the roots θ_i and the coefficients u_i and v_i.

Having chosen the sequences α_ℓ, β_ℓ and γ_ℓ in this way, we can indeed prove that $\mathcal{H} = O(n^{-3})$ in the whole of Ω. Furthermore, we show that $\mathcal{H}$ satisfies an additional 'continuity' condition that together with the condition $\mathcal{H} = O(n^{-3})$ will allow us to infer in the next sub-section that $\varepsilon = O(n^{-3})$.

Theorem 4.3

$$|\mathcal{H}_{n,k}| \le \frac{100}{n^3} \quad , \quad |\mathcal{H}_{n,k} - q^{(2)}_{n,k}\mathcal{H}_{n,k+2}| \le (1 - q^{(2)}_{n,k}) \cdot \frac{100}{n^3}$$

where $q^{(2)}_{n,k} = q_{n,k}q_{n,k+1}$, *for all positions in* Ω *with* $n \ge 1000$.

Proof: We first clarify the statement of the theorem. If $\mathcal{H} = (\mathcal{R}, \mathcal{S}, \mathcal{T})^T$ then we claim that $|\mathcal{R}_{n,k}|, |\mathcal{S}_{n,k}|, |\mathcal{T}_{n,k}| \le \frac{100}{n^3}$ for $n \ge 1000$ and $k \le \frac{2n+3}{3}, k \le \frac{2n+1}{3}, k \le \frac{2n+4}{3}$ respectively and that $|\mathcal{R}_{n,k} - q^{(2)}_{n,k}\mathcal{R}_{n,k+2}|, |\mathcal{S}_{n,k} - q^{(2)}_{n,k}\mathcal{S}_{n,k+2}|, |\mathcal{T}_{n,k} - q^{(2)}_{n,k}\mathcal{T}_{n,k+2}| \le (1 - q^{(2)}_{n,k})\frac{100}{n^3}$ for $n \ge 1000$ and $k \le \frac{2n-3}{3}, k \le \frac{2n-5}{3}, k \le \frac{2n-2}{3}$ respectively.

The rigorous proof of these inequalities is rather lengthy and technical. We shall only 'demonstrate' here the validity of two of them (those involving $\mathcal{R}_{n,k}$) using high level asymptotic analysis.

Assume at first that $k \le \frac{2n-2}{3}$ (the case $\frac{2n-1}{3} \le k \le \frac{2n+3}{3}$ will be dealt with separately). Consulting the definition of $(I - \Phi^*)$, we get that

$$\begin{aligned} \mathcal{R}_{n,k} &= -\mathcal{A}_{n,k} + p_{n,k}(1 + \mathcal{A}_{n-1,k-1}) - q_{n,k}\mathcal{B}_{n,k+1} \\ &= \left.-A_{n,k} + p_{n,k}(1 + A_{n-1,k-1}) - q_{n,k}B_{n,k+1} \quad \right\} = R_{n,k} \\ &\quad \left.-\tfrac{\alpha_{2n-3k}}{n^2} + p_{n,k} \cdot \tfrac{\alpha_{2n-3k+1}}{(n-1)^2} - q_{n,k} \cdot \tfrac{\beta_{2n-3k-3}}{n^2} \quad \right\} = \rho_{n,k} \end{aligned}$$

The term $R_{n,k}$ is a rational expression in n and k and automatic manipulations show that

$$R_{n,k} = -\tfrac{8-26\lambda^2+11\lambda^3}{16(2-\lambda)(1-\lambda)^5} \cdot \tfrac{1}{n^3} + O(\tfrac{1}{n^4}) \quad .$$

The coefficient of $\frac{1}{n^3}$ above attains its maximum absolute value in the range $[0, 2/3]$ at $\lambda \simeq 0.57$ where it evaluates to $\simeq -4.73$. We thus see that this term does not give us any cause for concern.

If we let $\ell = 2n - 3k$ we get

$$\begin{aligned}\rho_{n,k} &= \tfrac{1}{n^2}\left[-\alpha_\ell + p_{n,k}\cdot\tfrac{n^2}{(n-1)^2}\cdot\alpha_{\ell+1} - q_{n,k}\cdot\beta_{\ell-3}\right] \\ &= \tfrac{1}{n^2}\underbrace{\left[-\alpha_\ell + \frac{1}{2}\alpha_{\ell+1} - \frac{1}{2}\beta_{\ell-3}\right]}_{0} + \tfrac{1}{n^2}\left[\left(p_{n,k}\cdot\tfrac{n^2}{(n-1)^2} - \tfrac{1}{2}\right)\alpha_{\ell+1} - \left(q_{n,k} - \tfrac{1}{2}\right)\beta_{\ell-3}\right] \quad .\end{aligned}$$

The first expression in the last line disappears as it is one of the defining relations of the sequences $\{\alpha_\ell\}, \{\beta_\ell\}, \{\gamma_\ell\}$. We may assume now that $\ell = o(n)$ since otherwise $\alpha_{\ell+1}$ and $\beta_{\ell-3}$ are exponentially small and we have nothing to worry about. We can therefore use the relations

$$\begin{aligned} p_{n,k} &= \tfrac{2n-\ell}{4n+\ell} &&= \tfrac{1}{2} - \tfrac{3\ell}{8n} + o(\tfrac{1}{n}) \\ q_{n,k} &= \tfrac{2(n+\ell)}{4n+\ell} &&= \tfrac{1}{2} + \tfrac{3\ell}{8n} + o(\tfrac{1}{n}) \end{aligned}$$

together with the fact that $\frac{n^2}{(n-1)^2} = 1 + \frac{2}{n} + O(\frac{1}{n^2})$, to get that

$$\rho_{n,k} = \left[(1 - \tfrac{3\ell}{8})\alpha_{\ell+1} - \tfrac{3\ell}{8}\beta_{\ell-3}\right]\cdot\tfrac{1}{n^3} + o(\tfrac{1}{n^3}) \quad .$$

The coefficient of $\frac{1}{n^3}$ here is maximised when $\ell = 5$, where we get $\rho_{n,\frac{2n-4}{3}} \simeq -\frac{2.80}{n^3}$. Hence, for large enough n, and $k \le \frac{2n-2}{3}$ we even expect to have $|\mathcal{R}_{n,k}| \le \frac{10}{n^3}$.

Assume now that $\frac{2n-1}{3} \le k \le \frac{2n+3}{3}$. Proceeding in a similar way, we get that

$$\begin{aligned}\mathcal{R}_{n,k} &= -\mathcal{A}_{n,k} + p_{n,k}(1 + \mathcal{A}_{n-1,k-1}) \\ &= \left.-\mathcal{A}_{n,k} + p_{n,k}(1 + \mathcal{A}_{n-1,k-1}) - \tfrac{9(2n-3k-1)}{8n^2}\quad\right\} = R'_{n,k} \\ &\quad \left.-\tfrac{\alpha_{2n-3k}}{n^2} + p_{n,k}\cdot\tfrac{\alpha_{2n-3k+1}}{(n-1)^2} + \tfrac{9(2n-3k-1)}{8n^2}\quad\right\} = \rho'_{n,k}\end{aligned}$$

Again if $\ell = 2n - 3k$ then

$$R'_{n,k} \sim \tfrac{-135+198\ell-36\ell^2}{16}\cdot\tfrac{1}{n^3}$$

and the maximum of this expression in the range $-3 \le \ell \le 1$ is attained when $\ell = -3$ and $R'_{n,\frac{2n+3}{2}} \sim -\frac{65.8125}{n^3}$. As for $\rho'_{n,k}$ we get

$$\begin{aligned}\rho'_{n,k} &= \tfrac{1}{n^2}\left[-\alpha_\ell + p_{n,k}\cdot\tfrac{n^2}{(n-1)^2}\cdot\alpha_{\ell+1} + \tfrac{9(2n-3k-1)}{8}\right] \\ &= \tfrac{1}{n^2}\underbrace{\left[-\alpha_\ell + \tfrac{1}{2}\alpha_{\ell+1} + \tfrac{9(\ell-1)}{8}\right]}_{0} + \tfrac{1}{n^2}\left[\left(p_{n,k}\cdot\tfrac{n^2}{(n-1)^2} - \tfrac{1}{2}\right)\alpha_{\ell+1}\right] \\ &\sim \left(1 - \tfrac{3\ell}{8}\right)\alpha_{\ell+1}\cdot\tfrac{1}{n^3} \quad .\end{aligned}$$

The maximum absolute value is again attained when $\ell = -3$ where $\rho'_{n,\frac{2n+3}{2}} \simeq -\frac{9.91}{n^3}$. So, for large enough n, and any $k \le \frac{2n+3}{2}$, we expect to have $|\mathcal{R}_{n,k}| \le \frac{80}{n^3}$. The slackness that we have introduced by requiring only that $|\mathcal{R}_{n,k}| \le \frac{100}{n^3}$, allows us to prove this inequality for every $n \ge 1000$.

Turning our attention to the second inequality involving $\mathcal{R}_{n,k}$, we note that for $k \le \frac{2n-8}{3}$ we have

$$\frac{\mathcal{R}_{n,k} - q^{(2)}_{n,k}\mathcal{R}_{n,k+2}}{1 - q^{(2)}_{n,k}} = \frac{R_{n,k} - q^{(2)}_{n,k}R_{n,k+2}}{1 - q^{(2)}_{n,k}} + \frac{\rho_{n,k} - q^{(2)}_{n,k}\rho_{n,k+2}}{1 - q^{(2)}_{n,k}} \quad .$$

A simple manipulation yields

$$\frac{R_{n,k}-q^{(2)}_{n,k}R_{n,k+2}}{1-q^{(2)}_{n,k}} = R_{n,k} + \frac{q^{(2)}_{n,k}}{1-q^{(2)}_{n,k}}(R_{n,k} - R_{n,k+2}) \quad .$$

Note now that $R_{n,k} - R_{n,k+2} = O(n^{-4})$ or more precisely

$$R_{n,k} - R_{n,k+2} = \frac{-88+152\lambda+64\lambda^2-126\lambda^3+33\lambda^4}{16(2-\lambda)^2(1-\lambda)^6} \cdot \frac{1}{n^4} + O(\tfrac{1}{n^5}) \quad .$$

The coefficient of $\frac{1}{n^4}$ here is of course twice the derivative of the coefficient of $\frac{1}{n^3}$ in the corresponding expansion of $R_{n,k}$. It can be easily checked that $|R_{n,k}-R_{n,k+2}| \le \frac{3}{n^4}$ for say $\lambda \le \frac{1}{10}$. Now $q^{(2)}_{n,k}/(1-q^{(2)}_{n,k}) < 2n$ for every $k \ge 0$ and furthermore $q^{(2)}_{n,k}/(1-q^{(2)}_{n,k}) = O(1)$ whenever λ is bounded away from 0.

The term $(\rho_{n,k} - q^{(2)}_{n,k}\rho_{n,k+2})/(1-q^{(2)}_{n,k})$ attains a maximum of $\simeq -2.33/n^3$ for $\ell = 12$ and thus we can again obtain the desired inequality.

Combining these facts we get the desired bound for $k \le \frac{2n-8}{3}$. The case $\frac{2n-7}{3} \le k \le \frac{2n-3}{3}$ should again be treated separately. We omit the details.

The inequalities involving $\mathcal{S}_{n,k}$ and $\mathcal{T}_{n,k}$ could be 'verified' in a similar manner. $\square$

4.5 Bounding the errors

We saw in the previous sub-section that ε, the error of our estimation satisfies the equation $\varepsilon = \Phi^*\varepsilon + \mathcal{H}$ where $\mathcal{H}$ satisfies the conditions of Theorem 4.3. We now show that this implies $\varepsilon = O(n^{-3})$.

Theorem 4.4 *If $e = \Phi^* e + h$ where $e = (a,b,c)^T, h = (r,s,t)^T$ and*

$$|h_{n,k}| \le \frac{H}{n^3} \quad , \quad |h_{n,k} - q^{(2)}_{n,k}h_{n,k+2}| \le (1-q^{(2)}_{n,k}) \cdot \frac{H}{n^3}$$

for all positions in Ω with $n \ge n_0 \ge 1000$, and

$$|a_{n,k}|, |c_{n,k}| \le \frac{15H}{n^3} \quad , \quad |b_{n,k}| \le \frac{10H}{n^3}$$

for all positions in Ω with $n = n_0, n_0+1$, then the same bounds on $a_{n,k}, b_{n,k}$ and $c_{n,k}$ hold for all positions in Ω with $n \ge n_0$.

Proof : What conditions should two constants A and B satisfy if we are to succeed in proving by induction that $a_{n,k} \le \frac{A}{n^3}$ and that $b_{n,k} \le \frac{B}{n^3}$? Assuming the basis of the induction to be already established, we check what conditions on A and B enable us to derive the induction step.

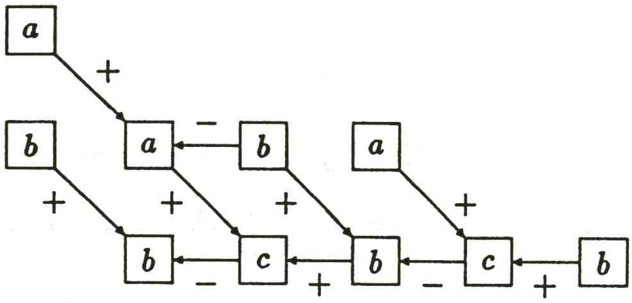

Figure 5: Expanding the definition of $b_{n,k}$.

Using the induction hypothesis and the conditions on $h_{n,k}$, we can bound $a_{n,k}$ as follows

$$\begin{aligned} |a_{n,k}| &\leq p_{n,k} \cdot |a_{n-1,k-1}| + q_{n,k} \cdot |b_{n,k+1}| + |r_{n,k}| \\ &\leq \left[\frac{p_{n,k}}{(n-1)^3}\right] \times A + \left[\frac{q_{n,k}}{n^3}\right] \times B + \left[\frac{1}{n^3}\right] \times H \quad . \end{aligned}$$

If $A > B + 2H$ then the last expression is less than $\frac{A}{n^3}$ for any sufficiently large n and $k \leq \frac{2n+3}{2}$. This is because in Ω we have $p_{n,k} \leq \frac{1}{2} + o(1)$.

In particular, if we choose $A = 15H, B = 10H$ we must verify that

$$\left[\frac{p_{n,k}}{(n-1)^3}\right] \times A + \left[\frac{q_{n,k}}{n^3}\right] \times B + \left[\frac{1}{n^3}\right] \times H \leq \frac{A}{n^3}$$

for any $n \geq 1000$ and $0 \leq k \leq 0.67n$. This inequality involves only quantities like $p_{n,k}$ and $q_{n,k}$ that were explicitly defined. Expanding these definitions we find that the claim to be verified is equivalent to the claim that

$$-2(4+3\lambda)(1-3n+3n^2) + (8-9\lambda)n^3 \geq 0$$

for any $n \geq 1000$ and $0 \leq \lambda = k/n \leq 0.67$. This inequality is easily verified.

The choice that $A > B$ has been to our advantage until now. It will however make our lives much more difficult in the sequel.

By expanding the recursive definitions of $a_{n,k}$,$b_{n,k}$ and $c_{n,k}$ in the way depicted in Fig. 5 we get that

$$\begin{aligned} b_{n,k} \quad = \quad & [p_{n,k}] \cdot b_{n-1,k-1} + [q^{(4)}_{n,k}] \cdot b_{n,k+4} \\ - \quad & [q_{n,k}(q_{n,k+1}p_{n,k+2} - p'_{n,k+1}q_{n-1,k})] \cdot b_{n-1,k+1} \\ - \quad & [q_{n,k}p^{(2)}_{n,k+1}] \cdot a_{n-2,k-1} + [q^{(3)}_{n,k}p'_{n,k+3}] \cdot a_{n-1,k+2} \\ - \quad & [q_{n,k}p^{(2)}_{n,k+1}] \cdot r_{n-1,k} + (s_{n,k} - [q^{(2)}_{n,k}] \cdot s_{n,k+2}) \\ - \quad & [q_{n,k}] \cdot (t_{n,k+1} - [q^{(2)}_{n,k+1}] \cdot t_{n,k+3}) \end{aligned}$$

where as before $q^{(2)}_{n,k} = q_{n,k}q_{n,k+1}$, $q^{(3)}_{n,k} = q^{(2)}_{n,k}q_{n,k+2}$, $q^{(4)}_{n,k} = q^{(3)}_{n,k}q_{n,k+3}$ and $p^{(2)}_{n,k} = p_{n,k}p_{n-1,k-1}$. We assume here that $k \le \frac{2n-11}{3}$ so that all the terms given are indeed present. The case $\frac{2n-10}{3} \le k \le \frac{2n+1}{3}$ must be dealt with separately and the details are omitted.

The important point to note here is the fact that $b_{n-1,k+1}$ contributes to $b_{n,k}$ along two different paths, once with a positive sign and again with a negative sign. Since $q_{n,k+1}p_{n,k+2} \approx p'_{n,k+1}q_{n-1,k}$ these two contributions almost cancel each other out. Thus, when we add up (the absolute values of) the coefficients of all the $a_{n',k'}$ and $b_{n',k'}$ appearing in this expansion for $b_{n,k}$ we get a quantity $\sigma_{n,k}$ which for $0 < \lambda$ is significantly less than 1. In fact, it is easy to check that $\sigma_{n,k} \sim \sigma(\lambda) = \frac{8-20\lambda+22\lambda^2-9\lambda^3}{(2-\lambda)^3}$. The function $\sigma(\lambda)$ attains the value $\frac{3}{4}$ at $\lambda = \frac{2}{3}$, the minimal value of $\frac{19}{27}(\simeq 0.704)$ at $\lambda = \frac{1}{2}$ and the maximal value of 1 at $\lambda = 1$. We see therefore that a choice $A, B \gg H$ should enable us to prove the induction step when λ is bounded away from 0 and n is large enough. We might expect trouble when $\lambda \simeq 0$ but this is exactly the place where the additional condition of Theorem 4.3 comes to our rescue. We have gone far enough in the expansion shown in Fig. 5 to obtain a configuration in which the driving terms tend to cancel each other in pairs.

Relying on the induction hypothesis and the conditions on $h_{n,k}$ we get that

$$\begin{aligned}
|b_{n,k}| \;\le\; & \left[\frac{q_{n,k}p^{(2)}_{n,k+1}}{(n-2)^3} + \frac{q^{(3)}_{n,k}p'_{n,k+3}}{(n-1)^3}\right] \times A \\
+ & \left[\frac{p_{n,k}}{(n-1)^3} + \frac{q_{n,k}(q_{n,k+1}p_{n,k+2} - p'_{n,k+1}q_{n-1,k})}{(n-1)^3} + \frac{q^{(4)}_{n,k}}{n^3}\right] \times B \\
+ & \left[\frac{q_{n,k}p^{(2)}_{n,k+1}}{n^3} + \frac{(1-q^{(2)}_{n,k})(1+q_{n,k})}{n^3}\right] \times H
\end{aligned}$$

where $q_{n,k+1}p_{n,k+2} - p'_{n,k+1}q_{n-1,k} = \frac{6(n-k-1)}{(2n-k-1)(2n-k-2)}$ is indeed positive for all relevant values of n and k.

We want to find values for A and B for which this last expression is less than or equal to $\frac{B}{n^3}$ for all large enough n and k in the appropriate range. Since we are not interested in finding the optimal constants A and B, we just point out that again the choice $A = 15H$ and $B = 10H$ suffices, i.e., the bound in the last inequality is less than $\frac{B}{n^3}$ for any $n \ge 1000$ and $k \le 0.67n$. Expanding again the definitions of $p_{n,k}, q_{n,k}$ and of $p^{(2)}_{n,k}, q^{(2)}_{n,k}, q^{(3)}_{n,k}, q^{(4)}_{n,k}$ we get that the condition that we have to verify is that the expression

$$\begin{aligned}
& 48\left(272 - 422\lambda + 141\lambda^2\right) \\
+ \; & 8\left(-9904 + 18558\lambda - 9863\lambda^2 + 1551\lambda^3\right) n \\
+ \; & 4\left(51824 - 115326\lambda + 81991\lambda^2 - 22071\lambda^3 + 1692\lambda^4\right) n^2 \\
+ \; & 2\left(-147888 + 391146\lambda - 350617\lambda^2 + 130959\lambda^3 - 18428\lambda^4 + 564\lambda^5\right) n^3 \\
+ \; & \left(247232 - 782580\lambda + 861216\lambda^2 - 415735\lambda^3 + 84636\lambda^4 - 5076\lambda^5\right) n^4 \\
+ \; & \left(-122384 + 473052\lambda - 630812\lambda^2 + 377241\lambda^3 - 100814\lambda^4 + 9306\lambda^5\right) n^5 \\
+ \; & \left(34392 - 170768\lambda + 275520\lambda^2 - 198141\lambda^3 + 65080\lambda^4 - 7857\lambda^5\right) n^6 \\
+ \; & \left(-4808 + 34920\lambda - 69032\lambda^2 + 58361\lambda^3 - 22308\lambda^4 + 3159\lambda^5\right) n^7 \\
+ \; & (2-\lambda)\left(112 - 1756\lambda + 3652\lambda^2 - 2590\lambda^3 + 567\lambda^4\right) n^8 \\
+ \; & \lambda(2-\lambda)\left(80 - 220\lambda + 178\lambda^2 - 39\lambda^3\right) n^9
\end{aligned}$$

is non-negative for $n \geq 1000$ and $0 \leq \lambda \leq 0.67$. To show that this is plausible we note that the function $\lambda(2-\lambda)(80-220\lambda+178\lambda^2-39\lambda^3)$ which is the coefficient of n^9 in the above expression is positive for $0 < \lambda \leq 0.67$. For values of λ close to 0 we also have to consider the coefficient of n^8 which is approximately 224 when $\lambda \simeq 0$. With slightly more technical work the positivity of this expression could be established rigorously.

Finally, for $c_{n,k}$ we get

$$|c_{n,k}| \leq p'_{n,k}|a_{n-1,k-1}| + q_{n,k}|b_{n,k+1}| \leq \frac{(15p'_{n,k}+10q_{n,k})H}{n^3} \leq \frac{12.5H}{n^3} \quad .$$

□

Theorem 4.5 *If $e=(a,b,c)^T$ is the solution of equation (1) (or (4)) and $\mathcal{E}=(\mathcal{A},\mathcal{B},\mathcal{C})^T$ is defined by (3),(5) and (6) then*

$$|a_{n,k}-\mathcal{A}_{n,k}| \leq \frac{1500}{n^3} \quad \text{for } n \geq 1000 \text{ and } 0 \leq k \leq \tfrac{2n+3}{2} \quad ,$$

$$|b_{n,k}-\mathcal{B}_{n,k}| \leq \frac{1000}{n^3} \quad \text{for } n \geq 1000 \text{ and } 0 \leq k \leq \tfrac{2n+1}{2} \quad ,$$

$$|c_{n,k}-\mathcal{C}_{n,k}| \leq \frac{1500}{n^3} \quad \text{for } n \geq 1000 \text{ and } 0 \leq k \leq \tfrac{2n+4}{2} \quad .$$

Proof : It can be verified directly that these inequalities hold for $n = 1000, /; 1001$ and all admissible values of k. The theorem then follows by combining Theorems 4.3 and 4.4. □

4.6 Beyond the boundary layer

We only have to consider the values of $a_{n,k}$ for $k \geq \frac{2n+4}{3}$. The values of $b_{n,k}$ for $k \geq \frac{2n+2}{3}$ are identically zero, by definition, and the values of $c_{n,k}$ for $k \geq \frac{2n+5}{3}$ are of no interest since they are never used. For $a_{n,k}$ in this region we have the simple relation

$$a_{n,k} = p_{n,k}(1+a_{n-1,k-1}) \quad \text{for } k \geq \tfrac{2n-1}{3} \quad .$$

By induction we can prove that for $k \geq \frac{2n-1}{3}$ we have

$$a_{n,k} = \frac{k}{2(n-k)+1} - \frac{k!}{(2n-k)!} \cdot \frac{[4(n-k)-2]!}{[2(n-k-1)]!} \cdot \left(\frac{2(n-k-1)}{2(n-k)+1} - a_{3(n-k)-2,2(n-k-1)}\right) \quad . \tag{7}$$

Note that the value $a_{3(n-k)-2,2(n-k-1)}$ lies in Ω, just outside the boundary layer as it is of the form $a_{n',\frac{2n'-2}{3}}$.

Using Stirling's formula we get that, for $k=\lambda n$ with $\frac{2}{3} < \lambda < 1$,

$$\frac{k!}{(2n-k)!} \cdot \frac{[4(n-k)-2]!}{[2(n-k-1)]!} \approx \sqrt{\frac{\lambda}{2-\lambda}} \cdot e^{-L(\lambda)n}$$

where

$$L(\lambda) = \ln\left[\frac{(2-\lambda)^{(2-\lambda)}}{\lambda^{\lambda}[8(1-\lambda)]^{2(1-\lambda)}}\right] \quad .$$

It can be checked that $L(2/3) = L(1) = 0$ while $L(\lambda) > 0$ for $\lambda \in (2/3, 1)$. Thus, for any k with $\lambda = k/n$ bounded away from both 2/3 and 1 we get that $a_{n,k} \simeq \frac{k}{2(n-k)+1}$ with an exponentially small error! The most accurate approximation is obtained for $k = \lambda n$ where $\lambda = 1 - \frac{1}{\sqrt{65}} \simeq 0.875965$ for which $L(\lambda) \simeq 0.249353$.

For $a_{n,n-\ell}$, equation (7) becomes

$$a_{n,n-\ell} = \frac{n-\ell}{2\ell+1} - \frac{(n-\ell)!}{(n+\ell)!} \cdot \frac{[2(2\ell-1)]!}{[2(\ell-1)]!} \cdot \left(\frac{2(\ell-1)}{2\ell+1} - a_{3\ell-2,2(\ell-1)}\right) \quad .$$

We can thus get explicit formulae for $a_{n,n-\ell}$ where ℓ is constant. All we have to know for this purpose is the single value of $a_{3\ell-2,2(\ell-1)}$. In particular we get

$$\begin{aligned} a_{n,n} &= n \\ a_{n,n-2} &= \frac{n-2}{5} - \frac{48}{(n+2)(n+1)n(n-1)} \\ a_{n,n-4} &= \frac{n-4}{9} - \frac{2983680}{(n+4)(n+3)\cdot\ldots\cdot(n-3)} \end{aligned}$$

and in general

$$a_{n,n-\ell} = \frac{n-\ell}{2\ell+1} + O(n^{-2\ell}) \quad .$$

Hence, the diagonals in the $e_{n,k}$ table behave essentially as linear sequences.

4.7 Verifying the optimal strategy

For $n \leq 1000$ the validity of the optimal strategy could be verified directly.

We now prove the validity of the optimal strategy for $n > 1000$ by induction. Suppose that we have already verified the claimed optimal strategy for all positions (n', k') with either $n' < n$ or $n' = n$ and $k' > k$. This means that so far, the values of the positions agree with those obtained from equation (1), and thus all the estimations of the previous sub-sections are valid. If $n + k$ is even and $k \neq 0, n$, we use these estimates to show that $e^1_{n,k} > 0, e^2_{n,k}$ (if $k = 0$ or n we already know that $e^1_{n,k} = e^2_{n,k}$). If $n + k$ is odd and $k \leq \frac{2n+1}{3}$ we use these estimates to show that $e^2_{n,k} > 0, e^1_{n,k}$, and if $n + k$ is odd and $k > \frac{2n+1}{3}$ we use them to show that $e^1_{n,k}, e^2_{n,k} < 0$. This will prove by induction the validity of the optimal strategy for every position.

As can be seen from Figures 2 and 3, the only inequalities for which we really need the $O(\frac{1}{n^2})$ terms in our approximations are those that claim that $e^2_{n,k} = b_{n,k} > 0$ when $n+k$ is odd and $k \leq \frac{2n+1}{3}$ and that $e^2_{n,k} < 0$ when $n + k$ is odd and $k \geq \frac{2n+2}{3}$. Even here, the $O(\frac{1}{n^2})$ terms are needed only when $\lambda \simeq \frac{2}{3}$.

5 Variants of the game

As was probably realised by the reader by now, there is no point in flipping back the cards after inspection if both players will remember them anyway. This convention also

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$n=1$	1	1						
$n=2$	0	$\frac{2}{3}$	2					
$n=3$	0	0	$\frac{1}{3}$	3				
$n=4$	0	$\frac{1}{7}$	$\frac{1}{3}$	0	4			
$n=5$	0	0	$\frac{1}{7}$	$\frac{4}{7}$	0	5		
$n=6$	0	$\frac{1}{231}$	$\frac{11}{105}$	$\frac{5}{42}$	$\frac{11}{14}$	0	6	
$n=7$	$\frac{151}{3003}$	$\frac{151}{3003}$	$\frac{20}{693}$	$\frac{272}{1155}$	$\frac{19}{210}$	$\frac{125}{126}$	0	7

Table 4: The expected values of the simplest positions when the 0- and 1-moves are allowed everywhere

allows the game to be played as a game of strategy by players with imperfect memories. A 0-move simply corresponds now to a decision to end the game while a 1-move will literally mean the inspection of one new card, without the useless ritual of inspecting an old one too. With these new conventions it seems natural to allow the 0-moves and 1-moves from all positions (even those of the form $(n,0)$ and $(n,1)$) and we shall do so throughout this section.

What is the effect of allowing the 0- and 1-moves from positions of the form $(n,0)$ and the 0-moves from positions of the form $(n,1)$? Since the value of every position in the new game is by definition non-negative some changes are bound to occur but as we shall soon see, the overall effect is minimal. The values of the simplest positions under the new rules are given in Table 4. These new values will of course influence the values of almost all other positions, but it turns out that the changes are exponentially diminishing in n for every $k = \lambda n$ with $\lambda < c < 1$. The new optimal moves from positions (n,k) with $k \le n \le 15$ are given in Table 5. There are again some exceptions when $n \le 5$ but, apart from that, the only difference between Table 5 and Table 2, giving the optimal moves in the original version of the game, is that a 0-move is now used from positions $(n,0)$ with n even. This was to be expected as the values of these positions were hitherto negative.

The analysis of this version of the game is almost identical to the one carried out here. The only difference is that a second boundary layer now exists when $\lambda \simeq 0$, caused by the 0-moves used from positions $(n,0)$ with n even.

We now turn to the study of variants of the game obtained by restricting the set of allowed moves. We have already encountered an example of this kind in sub-section 4.1 where we have assumed that the 0-moves are not allowed. We consider two other restricted versions.

$n = 1$	2 1
$n = 2$	0 2 1
$n = 3$	0 0 2 1
$n = 4$	0 2 1 0 1
$n = 5$	0 0 2 1 0 1
$n = 6$	0 2 1 2 1 0 1
$n = 7$	2 1 2 1 2 1 0 1
$n = 8$	0 2 1 2 1 2 1 0 1
$n = 9$	2 1 2 1 2 1 2 1 0 1
$n = 10$	0 2 1 2 1 2 1 2 1 0 1
$n = 11$	2 1 2 1 2 1 2 1 0 1 0 1
$n = 12$	0 2 1 2 1 2 1 2 1 0 1 0 1
$n = 13$	2 1 2 1 2 1 2 1 2 1 0 1 0 1
$n = 14$	0 2 1 2 1 2 1 2 1 2 1 0 1 0 1
$n = 15$	2 1 2 1 2 1 2 1 2 1 2 1 0 1 0 1

Table 5: The optimal moves for $n \leq 15$ when the 0- and 1-moves are allowed everywhere.

5.1 Version 1

In this sub-section we investigate the version of the game in which 1-moves are the only moves allowed. While there is no question of finding the optimal strategy in this case, the analysis of the expected gains from the different positions turns out to be interesting.

If we denote again by $e_{n,k}$ the expected gain from position (n, k), we get immediately the following recurrence relation

$$e_{n,k} = p_{n,k}(1 + e_{n-1,k-1}) - q_{n,k}e_{n,k+1} \tag{8}$$

where the only initial condition required is $e_{0,0} = 0$.

It turns out that in this version, each diagonal $e_{n,n-r}$ for a fixed r is an arithmetical progression,

$$e_{n,n-r} = \alpha_r n + \beta_r \tag{9}$$

say. By substituting this relation back into (8), we can prove (9) by induction and get that $\{\alpha_r\}$ and $\{\beta_r\}$ satisfy the relations

$$\begin{aligned} \alpha_r &= \tfrac{2r}{2r-1}\left(\tfrac{1}{2r} - a_{r-1}\right) & r \geq 1 \\ \beta_r &= \tfrac{1}{2}(\alpha_r - 1) - \beta_{r-1} & r \geq 1 \end{aligned}$$

where $\alpha_0 = 1, \beta_0 = 0$. By expanding the definition of α_r we get

$$\alpha_r = (-1)^r \tfrac{2\cdot 4\cdot\ldots\cdot 2r}{3\cdot 5\cdot\ldots\cdot(2r+1)}\left[1 - \tfrac{1}{2} + \tfrac{1\cdot 3}{2\cdot 4} - \tfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6} + \ldots + (-1)^r \tfrac{1\cdot 3\cdot\ldots\cdot(2r-1)}{2\cdot 4\cdot\ldots\cdot 2r}\right] \quad .$$

Recalling the Wallis product

$$\frac{\pi}{2} = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot \ldots}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot \ldots}$$

we have

$$\frac{2\cdot 4\cdot \ldots \cdot 2r}{3\cdot 5\cdot \ldots \cdot (2r+1)} \sim \sqrt{\frac{\pi}{2(2r+1)}} \quad .$$

The terms inside the square bracket have decreasing absolute values and alternating signs. By Leibnitz's Theorem the limit of the sum of this series exists as $r \to \infty$, and can be shown to be $\sqrt{2}/2$.

We conclude that

$$\alpha_r \sim \left(\frac{\pi}{8}\right)^{1/2} \frac{(-1)^r}{\sqrt{r}} \quad .$$

Similarly we can expand the definition of β_r and get that

$$\beta_r = \tfrac{1}{2}\cdot(\alpha_r - 1) - \tfrac{1}{2}\cdot(\alpha_{r-1} - 1) + \tfrac{1}{2}\cdot(\alpha_{r-2} - 1) - \ldots + \tfrac{1}{2}\cdot(-1)^r(\alpha_0 - 1)$$

or equivalently

$$\beta_r = \frac{1}{2}\cdot\sum_{s=0}^{r}(-1)^{r-s}\alpha_s - \begin{cases} 0 & \text{if } r \text{ odd} \\ \frac{1}{2} & \text{otherwise} \end{cases} \quad .$$

Since $\alpha_r \sim (\pi/8)^{1/2}(-1)^r r^{-1/2}$, we derive immediately that

$$\beta_r \sim \left(\tfrac{\pi}{8}\right)^{1/2}(-1)^r\sqrt{r} \quad .$$

Hence, if $n - k \to \infty$ then

$$e_{n,k} = \alpha_{n-k}n + \beta_{n-k} \sim \left(\tfrac{\pi}{8}\right)^{1/2}(-1)^{n-k}\left[\tfrac{n}{\sqrt{n-k}} + \sqrt{n-k}\right]$$

and, in particular,

$$e_{n,0} \sim \left(\tfrac{\pi}{2}\right)^{1/2}(-1)^n\sqrt{n} \quad .$$

So the behaviour of the $e_{n,k}$, as well as the methods used to obtain them are quite different in this case.

5.2 Version 2

In this sub-section we check what happens if the 2-moves are the only moves allowed. The analysis in this case can serve as an introductory example to the use of the bootstrapping method of sub-section 4.3. We omit the details but point out that, in contrast to what we have seen so far, the parity of $n + k$ does not play a major role, unless $\lambda = k/n \simeq 1$. The asymptotic expansion for $e_{n,k}$ obtained by bootstrapping is

$$e_{n,k} = \tfrac{\lambda^2}{4(2-\lambda)(1-\lambda)} + \tfrac{16-64\lambda+64\lambda^2-19\lambda^3}{16(2-\lambda)^2(1-\lambda)^2}\cdot\tfrac{1}{n} + \tfrac{64-144\lambda+216\lambda^2-198\lambda^3+69\lambda^4}{64(2-\lambda)^3(1-\lambda)^3}\cdot\tfrac{1}{n^2} + \ldots$$

and it is again valid whenever λ is bounded away from 1.

6 More possibilities

In this paper we investigated the two-player game assuming that both players have perfect memory and that their objective is to maximise their expected gain. What happens if the players have imperfect memories?

What happens if the objective of the players is just to win the game? This was investigated by the first author and the results may appear in a planned subsequent paper. A position is now characterised by a triplet (n, k, ℓ) where ℓ is the current difference between the number of pairs held by the two players. It turns out that when a player is in the lead, or at least even (i.e. $\ell \geq 0$), her optimal strategy is roughly the same as the gain maximising strategy. If, on the other hand, a player is trailing behind, then she has no choice but to take more chances and play 2-moves whenever 0-moves are suggested by the gain optimising strategy.

What happens if more players join the game? This was again investigated by the first author and the results may again appear in a subsequent paper. As an 'appetizer' we just state that, if the objective is to maximise the expected gain, then the optimal move from position (n, k) in the three-player game is a 0-move if $n - k \equiv 2$ and $k \geq 3$, a 1-move if $n - k \equiv 0$ and $k \geq 1$, and a 2-move otherwise, that is, if $n - k \equiv 1$ or $k = 0$ or $k = 1$ and $n \equiv 0$. All these congruences are modulo 3.

The game is sometimes played with a deck of cards composed of quadruples instead of pairs. What is then the optimal strategy?

What is the effect of allowing the players to inspect more than two cards in their attempts to find pairs?

7 Concluding remarks

The optimal strategy for playing the memory game turns out to be very simple. The analysis and proof presented here were however extremely involved. Is there an easier way of proving the results stated in Section 3?

While the results of this work are mainly of recreational value, we hope that the methods used here will prove useful elsewhere. We would like to stress again the indispensable role played in this work by experimentation and by automatic symbolic computations.

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